4.5 – Coordinates and Basis

Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a **basis** for V if:

a) S spans V.

b) S is linearly independent.

#1 Use the determinant of a coefficient matrix to show that the following set of vectors forms a basis for R^2: $\{(2,\ 1),\ (3,\ 0)\}$.

#8 Show that the following vectors do not form a basis for P_2.
$1 - 3x + 2x^2$, $1 + x + 4x^2$, $1 - 7x$

Definition: A basis in which the listed order of the vectors matters is called an **ordered basis**.

Some Standard bases

$\{\hat{i},\ \hat{j}\}$ is a basis for R^2 (this is the same as $\{\mathbf{e}_1, \mathbf{e}_2\}$).
$\{\hat{i},\ \hat{j},\ \hat{k}\}$ is basis for R^3 (this is the same as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$).
$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis for R^n.
$\{1,\ x,\ x^2,\ \dots,\ x^n\}$ is a basis for P_n.
$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for M_{22}.

Theorem 4.5.1 Uniqueness of Basis Representation
If $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + \dots + c_n\,\mathbf{v}_n$ in exactly one way.

Definition: If $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_n\}$ is an ordered basis for a vector space V, and $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$ is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1,\ c_2,\ \dots,\ c_n$ are called the **coordinates of v relative to the basis** S. The vector $(c_1,\ c_2,\ \dots,\ c_n)$ in R^n constructed from these coordinates is called the **coordinate vector of v relative to the basis** S; it is denoted by $(\mathbf{v})_S = (c_1,\ c_2,\ \dots,\ c_n)$.

Example: Consider $\mathbf{v} = (-1,\ 7,\ 2) \in R^3$. The set $B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$, where $\mathbf{v}_1 = (2,\ -1,\ -1)$, $\mathbf{v}_2 = (-2,\ 1,\ 2)$, and $\mathbf{v}_3 = (3,\ 5,\ 4)$ forms a basis for R^3 (verify). Find $(\mathbf{v})_S$ (where S is the standard basis for R^3) and $(\mathbf{v})_B$.

#16 First show that the set $S = \{A_1,\ A_2,\ A_3,\ A_4\}$ is a basis for M_{22}, then express A as a linear combination of the vectors in S, and then find the coordinate vector of A relative to S.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};\ A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$$